Math 275D Lecture 6 Notes

Daniel Raban

October 9, 2019

1 Markov Property of Brownian Motion

1.1 Proof of the Markov property of Brownian motion

We want to prove the following.

Lemma 1.1 (Markov property of Brownian motion). *Let $Y : C(\mathbb{R}) \to [-M, M]$ be a functional, and let θ_s be the shift by s. Then*

$$\mathbb{E}_x[Y \circ \theta_s \mid \mathcal{F}_s^+] = \mathbb{E}_{B(s)}[Y].$$

Recall the following lemma:

Theorem 1.1 (Monotone class argument). *Let $\mathcal{A}$ be a π-system, and let $\mathcal{H}$ be a collection of functions that satisfies:*

1. *If $A \in \mathcal{A}$, then $1_A \in \mathcal{H}$.*
2. *If $f, g \in \mathcal{H}$, then $f + g \in \mathcal{H}$.*
3. *If $f_n \uparrow f$ with $0 \leq f_n \in \mathcal{H}$, then $f \in \mathcal{H}$.*

Then $\mathcal{H}$ has all bounded measurable functions with respect to $\sigma(\mathcal{A})$.

Now on to the proof of the Markov property.

Proof. We need to show that for any $A \in \mathcal{F}_s^+$ and for any Y, $\mathbb{E}[\mathbb{1}_A(Y \circ \theta_s)] = \mathbb{E}[\mathbb{1}_A \, \mathbb{E}_{B(s)}[Y]]$. By the monotone class argument, we only need to prove this for Ys of the form $Y = \prod_{k=1}^n f_k(B(t_k))$; here, $\mathcal{A}$ is the algebra generated by the $B(t)$s. Then

$$Y \circ \theta_s(B(\cdot)) = \prod_{k=1}^n f_k(B(t_k + s)).$$

The point of this trick now is that the difference between $\mathcal{F}_s^0$ and $\mathcal{F}_s^+$ doesn't matter because there is a gap between s and $s + t_1$.

For this Y, let's prove that that the result holds for any $A \in \mathcal{F}^0_{S+\delta}$ with $\delta \le t_1/2$. So we want to show that

$$\mathbb{E}[Y \circ \theta_s \mid \mathcal{F}^0_{s+\delta}] = \varphi^Y_{B(s+\delta)}$$

for some φ. We only need to show that this holds for $A \in \mathcal{A}_\delta$, where $\mathcal{A}_\delta = \{\bigcap_{k=1}^n \{B(a_k) \in R_k\} : a_k \in [0, s+\delta], R_k \text{ is Borel}\}$, because $\sigma(\mathcal{A}_\delta) = \mathcal{F}^0_{s+\delta}$. So we can calculate

$$\begin{aligned}
\mathbb{E}[1_A(Y \circ \theta_s)] &= \mathbb{E}\left[\prod_{k=1}^n 1_{\{B(a_k)\in R_k\}} \cdot \prod_{k=1}^n f_k(B(t_k+s))\right] \\
&= \int_{R_1} p_{a_1}(0, \hat{a}_1)\, d\hat{a}_1 \int_{R_2} p_{a_2-a_1}(\hat{a}_1, \hat{a}_2)\, d\hat{a}_2 \cdots \int_{R_n} p_{a_n-a_{n-1}}(\hat{a}_{n-1}, \hat{a}_n)\, d\hat{a}_n \\
&\quad \cdot \int_R p_{s+\delta-a_n}(\hat{a}_n, \xi)\, d\xi \int_R p_{t-\delta}(\xi, \hat{t}_1) f_1(\hat{t}_1)\, dt_1 \cdot \int_R p_{t_2-t_1}(\hat{t}_1, \hat{t}_2) f_2(\hat{t}_2)\, d\hat{t}_2 \\
&\quad \cdots \int_R p_{t_m-t_{m-1}}(\hat{t}_{m-1}, \hat{t}_m) f_m(\hat{t}_m) d\hat{t}_m,
\end{aligned}$$

where $p_t(x,y)$ is the pdf of a $N(0,t)$ random variable, $p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-|y-x|^2/2t}$. These integrals should be nested (evaluate them in reverse order).

$$= \mathbb{E}[\mathbb{1}_A \varphi^Y_{s+\delta}],$$

where $\varphi^Y_{s+\delta} \in \mathcal{F}^0_{s+\delta}$. In particular, this depends only on $B(s+\delta)$. If we send $\delta \downarrow 0$, this φ has a limit. This limit is exactly $\varphi^Y(B_s) = \mathbb{E}_{B(s)}[Y]$. □

1.2 Events at 0 and ∞ are trivial

Corollary 1.1. *$\mathcal{F}^+_0$ is trivial.*

Proof. $\mathcal{F}^+_0$ agrees with $\mathcal{F}^0_0$ mod null sets, but $\mathcal{F}^0_0$ is trivial. □

Corollary 1.2. *Events depending on $B(t)$ as $t \to \infty$ are trivial.*

Proof. Define $Y(t) = tB(1/t)$. Then $Y(t)$ is a Brownian motion because they are both have finite dimensional distributions which are Gaussian vectors that agree. But events with $t \to \infty$ are the same as events for $B(s)$ with $s \downarrow 0$. These are trivial. □